

# LOCALIZATION OF EQUIVARIANT COHOMOLOGY - INTRODUCTORY AND EXPOSITORY REMARKS

A.A. BYTSENKO AND F.L. WILLIAMS

## 1. INTRODUCTION

In 1982 J.J. Duistermaat and G. Heckman [12] found a formula which expressed certain oscillatory integrals over a compact symplectic manifold as a sum over critical points of a corresponding phase function. In this sense these integrals are localized, and their stationary-phase approximation is exact with no error terms occurring. The ideas and techniques of localization extended to infinite-dimensional settings have proved to be quite useful and indeed central for many investigations in theoretical physics - investigations ranging from supersymmetric quantum mechanics, topological and supersymmetric field theories, to integrable models and low-dimensional gauge theories, including two-dimensional Yang-Mills theory [25]. Path integral localization appears in the work of M. Semenov-Tjan-Schanskii [23], which actually predates [12].

E. Witten was the first to propose an extension of the Duistermaat-Heckman (D-H) formula to an infinite-dimensional manifold - namely to the loop space  $LM$  of smooth maps from the circle  $S^1$  to a compact orientable manifold  $M$ . In this case a purely formal application of the D-H formula to the partition function of  $N = 1/2$  supersymmetric quantum mechanics yields a correct formula for the index of a Dirac operator [1]. Further arguments in this direction were presented with mathematical rigor by J.-M. Bismut in [7, 8].

The various generalizations of D-H generally require formulations in terms of equivariant cohomology. One has, for example, the Berline-Vergne (B-V) localization formula [3, 4, 5, 6] which expresses the integral of an equivariant cohomology class as a sum over zeros of a vector field to which that class is related; also see [9, 16, 25, 29] for example, a broader formulation of the localization formula. Our remarks here are designed to provide members of the Conference, and others, with

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a brief introduction to the B-V localization formula, and to indicate how the D-H formula is derived from it. Thus our goal is deliberately very modest. We shall limit our discussion, in particular, to the finite-dimensional setting as our idea is to convey the basic flavor of these formulas. This introduction should prepare readers for quite more ambitious discussions found in [6, 16, 25], for example.

The role of equivariant cohomology in physical theories will continue to grow as it has grown in past years. In particular it will be an indispensable tool for topological theories of gauge, strings, and gravity. We thank the organizers of this Conference for this opportunity to present these brief remarks on a topic of such growing interest in the physics community.

## 2. THE EQUIVARIANT COHOMOLOGY SPACE $H(M, X, s)$

For an integer  $j \geq 0$  let  $\Lambda^j M$  denote the space of smooth complex differential forms of degree  $j$  on a smooth manifold  $M$ .  $d : \Lambda^j M \rightarrow \Lambda^{j+1} M$  will denote exterior differentiation, and for a smooth vector field  $X$  on  $M$ ,  $\theta(X) : \Lambda^j M \rightarrow \Lambda^j M$ ,  $i(X) : \Lambda^j M \rightarrow \Lambda^{j-1} M$  will denote Lie and interior differentiation by  $X$ , respectively:

$$\begin{aligned} (\theta(X)\omega)(X_1, \dots, X_j) &= X\omega(X_1, \dots, X_j) \\ &- \sum_{\ell=1}^j \omega(X_1, \dots, X_{\ell-1}, [X, X_\ell], X_{\ell+1}, \dots, X_j), \end{aligned} \quad (2.1)$$

$$(i(X)\omega)(X_1, \dots, X_{j-1}) = \omega(X, X_1, \dots, X_{j-1}) \quad (2.2)$$

for  $\omega \in \Lambda^j M$  and for  $X_1, \dots, X_j \in VM$  = the space of smooth vector fields on  $M$ . One has the familiar rules

$$\begin{aligned} \theta(X) &= di(X) + i(X)d, \\ d\theta(X) &= \theta(X)d, \quad \theta(X)i(X) = i(X)\theta(X), \\ i(X)^2 &= 0; \quad \text{of course } d^2 = 0. \end{aligned} \quad (2.3)$$

For a complex number  $s$  let

$$d_{X,s} = d + si(X) \quad \text{on } \Lambda M = \sum_{j \geq 0} \oplus \Lambda^j M. \quad (2.4)$$

Then by (2.3),  $d_{X,s}\theta(X) = \theta(X)d_{X,s}$  and  $d_{X,s}^2 = s\theta(X)$ . Hence the subspace

$$\Lambda_X M = \{\omega \in \Lambda M \mid \theta(X)\omega = 0\}, \quad (2.5)$$

of  $\Lambda M$  is  $d_{X,s}$ -invariant and  $d_{X,s}^2 = 0$  on  $\Lambda_X M$ . It follows that we can define the cohomology space

$$H(M, X, s) = Z(M, X, s)/B(M, X, s) \quad (2.6)$$

for  $Z(M, X, s) = \text{kernel of } d_{X,s} \text{ on } \Lambda_X M$ ,  $B(M, X, s) = d_{X,s}\Lambda_X M$ . The space  $H(M, X, s)$  appears to depend on the parameter  $s$ . However it is not difficult to show that for  $s \neq 0$  there is an isomorphism of  $H(M, X, s)$  onto  $H(M, X, 1)$ . For  $X = 0$ ,  $H(M, 0, s)$  is the ordinary de Rham cohomology of  $M$ .

We shall be interested in the case when  $M$  has a smooth Riemannian structure  $\langle, \rangle$ , and when  $M$  is oriented and even-dimensional. Thus let  $\omega \in \Lambda^{2n} M - \{0\}$ ,  $\dim M = 2n$ , define the orientation of  $M$ . In this case we assume moreover that  $X$  is a Killing vector field:

$$X \langle X_1, X_2 \rangle = \langle [X, X_1], X_2 \rangle + \langle X_1, [X, X_2] \rangle \quad (2.7)$$

for  $X_1, X_2 \in VM$ . If  $p \in M$  is a zero of  $X$  (i.e.  $X_p = 0$ ) then there is an induced linear map  $\mathfrak{L}_p(X)$  of the tangent space  $T_p(M)$  of  $M$  at  $p$  such that

$$\mathfrak{L}_p(X)(Z_p) = [X, Z]_p \quad \text{for } Z \in VM. \quad (2.8)$$

Because of (2.7) one has that  $\mathfrak{L}_p(X)$  is skew-symmetric; i.e.  $\langle \mathfrak{L}_p(X)V_1, V_2 \rangle_p = -\langle V_1, \mathfrak{L}_p(X)V_2 \rangle_p$  for  $V_1, V_2 \in T_p(M)$ . Let  $f_p(X) : T_p(M) \oplus T_p(M) \rightarrow \mathbb{R}$  be the corresponding skew-symmetric bilinear form on  $T_p M$ :

$$f_p(X)(V_1, V_2) = \langle V_1, \mathfrak{L}_p(X)V_2 \rangle_p \quad \text{for } V_1, V_2 \in T_p M. \quad (2.9)$$

In order to apply some standard linear algebra to the real inner product space  $(T_p(M), \langle, \rangle_p)$ , we suppose  $\mathfrak{L}_p(X)$  is a non-singular linear operator on  $T_p(M) : \det \mathfrak{L}_p(X) \neq 0$ ; equivalently, this means that the bilinear form  $f_p(X)$  is non-degenerate. Then one can find an ordered orthonormal basis  $e = e^{(p)} = \{e_j = e_j^{(p)}\}_{j=1}^{2n}$  of  $T_p(M)$  such that

$$\begin{aligned} \mathfrak{L}_p(X)e_{2j-1} &= \lambda_j e_{2j}, \\ \mathfrak{L}_p(X)e_{2j} &= -\lambda_j e_{2j-1}, \quad \text{for } 1 \leq j \leq n, \end{aligned} \quad (2.10)$$

where each  $\lambda_j \in \mathbb{R} - \{0\}$ . In other words, relative to  $e$  the matrix of  $\mathfrak{L}_p(X)$  has the form

$$\mathfrak{L}_p(X) = \begin{bmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 & -\lambda_n \\ & & & & \lambda_n & 0 \end{bmatrix}. \quad (2.11)$$

Moreover, interchanging  $e_1, e_2$  if necessary, we can assume that  $e$  is positively oriented:  $\omega_p(e_1, \dots, e_{2n}) > 0$ . Finally, consider the Pfaffian  $\text{Pf}_e(\mathfrak{L}_p(X))$  of  $\mathfrak{L}_p(X)$  relative to  $e$ :

$$\text{Pf}_e(\mathfrak{L}_p(X)) = \frac{1}{n!} [f_p(X) \wedge \dots \wedge f_p(X)](e_1, \dots, e_{2n}). \quad (2.12)$$

$\text{Pf}_e(\mathfrak{L}_p(X))$  satisfies

$$(*) \quad \text{Pf}_e(\mathfrak{L}_p(X))^2 = \det \mathfrak{L}_p(X),$$

$$(**) \quad \text{Pf}_e(\mathfrak{L}_p(X)) = (-1)^n \lambda_1 \cdots \lambda_n.$$

If  $e' = \{e'_j\}_{j=1}^{2n}$  is another ordered, positively oriented orthogonal basis of  $T_p(M)$  then

$$\text{Pf}_{e'}(\mathfrak{L}_p(X)) = \text{Pf}_e(\mathfrak{L}_p(X)). \quad (2.13)$$

Equation (2.13) means that we can define a square-root of  $\mathfrak{L}_p(X)$  by setting

$$[\det \mathfrak{L}_p(X)]^{1/2} = (-1)^n \text{Pf}_e(\mathfrak{L}_p(X)). \quad (2.14)$$

That is, the square-root is independent of the choice  $e$  of an ordered, positively oriented orthogonal basis of  $T_p(M)$ . By  $(**)$  we have  $[\det \mathfrak{L}_p(X)]^{1/2} = \lambda_1 \cdots \lambda_n$ . The reader is reminded that the hypotheses  $X_p = 0$  and  $\det(\mathfrak{L}_p(X)) \neq 0$  were imposed, with  $X$  a Killing vector field.

### 3. THE LOCALIZATION FORMULA

As before we are given an oriented,  $2n$ -dimensional Riemannian manifold  $(M, \omega, \langle, \rangle)$ . Now assume that  $G$  is a compact Lie group which acts smoothly on  $M$ , say on the left, and that the metric  $\langle, \rangle$  is  $G$ -invariant. Let  $\mathfrak{g}$  denote the Lie algebra of  $G$ . Given  $X \in \mathfrak{g}$ , there is an induced vector field  $X^* \in VM$  on  $M$ : for  $\phi \in C^\infty(M)$ ,  $p \in M$

$$(X^* \phi)(p) = \frac{d}{dt} \phi(\exp(tX) \cdot p)|_{t=0}. \quad (3.1)$$

Since  $\langle, \rangle$  is  $G$ -invariant, one knows that  $X^*$  is a Killing vector field.  $X^*$  is said to be non-degenerate if for every zero  $p \in M$  of  $X^*$ , the induced linear map  $\mathfrak{L}_p(X^*) : T_p(M) \rightarrow T_p(M)$  non-singular. Since  $X^*$  is a Killing vector field,  $\mathfrak{L}_p(X^*)$  is skew-symmetric with respect to the inner product structure  $\langle, \rangle_p$  on  $T_p(M)$ , as we have noted, and non-singularity of  $\mathfrak{L}_p(X^*)$  means that we can construct the square-root

$$[\det \mathfrak{L}_p(X^*)]^{1/2} = (-1)^n \text{Pf}_e(\mathfrak{L}_p(X^*)) = \lambda_1 \cdots \lambda_n, \quad (3.2)$$

as in (2.14).

For a form  $\tau \in \Lambda M = \sum \oplus \Lambda^j M$  we write  $\tau_j \in \Lambda^j M$  for its homogeneous  $j$ -th component,

$$\tau = (\tau_0, \dots, \tau_{2n}) = \sum_{j=0}^{2n} \tau_j, \quad (3.3)$$

and we write  $[\tau]$  for the cohomology class of  $\tau$  in case  $\tau \in Z(M, Y, s)$  for  $Y \in VM$ ,  $s \in \mathbb{C}$ ; i.e.  $d_{Y,s}\tau = 0$  for  $d_{Y,s}$  in (2.4). When  $M$  is compact, in particular, one can integrate any  $2n$ -form (as  $M$  is orientable). Thus we can define

$$\int_M \tau = \int_M \tau_{2n}, \quad (3.4)$$

and in fact we can define

$$\int_M [\tau] = \int_M \tau = \int_M \tau_{2n}. \quad (3.5)$$

The integral  $\int_M [\tau]$  really does depend only on the class  $[\tau]$  of  $\tau$ . That is, if  $\tau' \in B(M, Y, s)$  then by a quick computation using Stokes' theorem one sees that  $\int_M \tau' \stackrel{(i)}{=} 0$ . Similarly if  $p \in M$  with  $Y_p = 0$  then  $\tau'_0(p) = 0$  for  $\tau' \in B(M, Y, s)$ . In fact if we write  $\tau' = d_{Y,s}\beta$  for  $\beta \in \Lambda_Y M$  then one has

$$\begin{aligned} \tau' &= (si(Y)\beta_1, d\beta_0 + si(Y)\beta_2, d\beta_1 + si(Y)\beta_3, d\beta_2 + si(Y)\beta_4, \\ &\quad \dots, d\beta_{2n-2} + si(Y)\beta_{2n}, d\beta_{2n-1}) \\ &= d\beta_0 + si(Y)\beta_0 + d\beta_1 + si(Y)\beta_1 + d\beta_2 + si(Y)\beta_2 \\ &\quad + \dots + d\beta_{2n} + si(Y)\beta_{2n}. \end{aligned} \quad (3.6)$$

Thus  $\tau'_0(p) = s\beta_{1p}(Y_p) = 0$ , and  $\int_M \tau' = \int_M d\beta_{2n-1} = 0$ , which proves (i). It follows that the map  $p^* : H(M, Y, s) \rightarrow \mathbb{R}$  given by

$$p^*[\tau] = \tau_0(p) \text{ for } Y_p = 0 \quad (3.7)$$

is well-defined.

In [3, 4, 5], N. Berline and M. Vergne, following some ideas of R. Bott in [10], established the following localization theorem, where the choice  $s = -2\pi\sqrt{-1}$  is made.

**Theorem 3.1.** *Assume as above that  $M$  and  $G$  are compact and that the Riemannian metric  $\langle, \rangle$  on  $M$  is  $G$ -invariant; i.e. each  $a \in G$  acts as an isometry of  $M$ . For  $X \in \mathfrak{g}$ , the Lie algebra of  $G$ , assume that the induced vector field  $X^*$  on  $M$  (see (3.1)) is non-degenerate; thus the square-root in (3.2) is well-defined (and is non-zero) for  $p \in M$  a zero of  $X^*$  (i.e.  $X_p^* = 0$ ). Then for any cohomology class  $[\tau] \in H(M, X^*, -2\pi\sqrt{-1})$  one has*

$$\int_M [\tau] = (-1)^{n/2} \sum_{\substack{p \in M, \\ p = \text{a zero of } X^*}} \frac{p^*[\tau]}{[\det \mathfrak{L}_p(X^*)]^{1/2}}, \quad (3.8)$$

see (3.5), (3.7).

For concrete applications of Theorem 3.1 we shall need to construct concrete cohomology classes in  $H(M, X^*, -2\pi\sqrt{-1})$ . The construction of such classes requires that a bit more be assumed about  $M$  and  $G$ . Suppose for example that  $M$  has a symplectic structure  $\sigma : \sigma \in \Lambda^2 M$  is a closed 2-form (i.e.  $d\sigma = 0$ ) such that for every  $p \in M$  the corresponding skew-symmetric form  $\omega_p : T_p(M) \oplus T_p(M) \rightarrow \mathbb{R}$  is non-degenerate. In particular  $M$  is oriented by the Liouville form

$$\omega_\sigma = \frac{1}{n!} \sigma \wedge \cdots \wedge \sigma \in \Lambda^{2n} M - \{0\}. \quad (3.9)$$

Suppose also that there is a map  $J : \mathfrak{g} \rightarrow C^\infty(M)$  which satisfies

$$i(X^*)\sigma + dJ(X) = 0, \quad \forall X \in \mathfrak{g}, \quad (3.10)$$

an equality of 1-forms. The existence of such a map  $J$  amounts to the assumption that the action of  $G$  on  $M$  is Hamiltonian, a point which we shall return to later. Given  $J$  define for each  $X \in \mathfrak{g}$  the form  $\tau^X \in \Lambda M$  by

$$\tau^X = \left( J(X), 0, -\frac{\sigma}{2\pi\sqrt{-1}}, 0, \dots, 0 \right); \quad (3.11)$$

see (3.3). We claim that  $\tau^X \in Z(M, X^*, -2\pi\sqrt{-1})$ . Since  $J(X)$  is a function  $i(X^*)J(X) = 0$ . Therefore by (2.3) and (3.10),  $\theta(X^*)J(X) = i(X^*)dJ(X) = -i(X^*)^2\sigma = 0$  and  $\theta(X^*)\sigma = di(X^*)\sigma + i(X^*)d\sigma = di(X^*)\sigma$  (as  $d\sigma = 0$ )  $= -d^2J(X) = 0$ . By definition (3.11) it follows that  $\theta(X^*)\tau^X = (\theta(X^*)J(X), 0, -\theta(X^*)\sigma/2\pi\sqrt{-1}, 0, \dots, 0) = 0$ , which

by (2.5) means that  $\tau^X \in \Lambda_{X^*}M$ . Also for  $s = -2\pi\sqrt{-1}$ , by definition (2.4) and (3.10),  $d_{X^*,s}\tau^X = (d + si(X^*))\tau^X = dJ(X) + si(X^*)J(X) - d\sigma/2\pi\sqrt{-1} - si(X^*)\sigma/2\pi\sqrt{-1} = -i(X^*)\sigma + i(X^*)\sigma = 0$ , which verifies the claim, where again we have used that  $i(X^*)J(X) = 0$ ,  $d\sigma = 0$ . Thus, given  $J$ , we have for each  $X \in \mathfrak{g}$  a cohomology class  $[\tau^X] \in H(M, X^*, -2\pi\sqrt{-1})$ .

#### 4. THE CLASS $[e^{c\tau^X}]$

In the next section the Duistermaat-Heckman formula will be derived by a direct application of Theorem 3.1. The main point is the construction of an appropriate cohomology class. Namely for the cocycle  $\tau^X \in Z(M, X^*, -2\pi\sqrt{-1})$  in (3.11) we wish to construct for  $c \in \mathbb{C}$  a well-defined form  $e^{c\tau^X}$  which also is an element of  $Z(M, X^*, -2\pi\sqrt{-1})$ .

Thus again suppose  $J$  which satisfies (3.10) is given. For  $X \in \mathfrak{g}$  let  $\tau_0 = J(X)$ ,  $\tau_1 = 0$ ,  $\tau_2 = -\sigma/2\pi\sqrt{-1}$ ,  $\tau_j = 0$  for  $3 \leq j \leq 2n$ , and let  $\tau = \tau^X$ . That is, by (3.11),  $\tau = (\tau_0, \tau_1, \tau_2, \dots, \tau_{2n}) = (\tau_0, 0, \tau_2, 0, 0, \dots, 0)$ . If  $\omega_1, \omega_2$  are forms of degree  $p, q$  respectively, then  $\omega_1$  and  $\omega_2$  commute if either  $p$  or  $q$  is even, since  $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$ . In particular  $\tau_0$  and  $\tau_2$  commute. Now if  $A$  and  $B$  are commuting matrices one has  $e^{A+B} = e^A \cdot e^B$ . Since  $\tau_0$  and  $\tau_2$  commute we should have, formally for any complex number  $c$ ,  $c\tau = c\tau_0 + c\tau_2 \Rightarrow e^{c\tau} = e^{c\tau_0} \cdot e^{c\tau_2} = e^{c\tau_0}(1 + c\tau_2 + c^2\tau_2^2/2! + c^3\tau_2^3/3! + \dots)$ , with  $\tau_2^j = \tau_2 \wedge \dots \wedge \tau_2$  ( $j$  times)  $\in \Lambda^{2j}M$ . Since  $\Lambda^{2j}M = 0$  for  $j > n$  we can take  $\sum_{j=0}^{\infty} c^j \tau_2^j / j!$  to mean  $\sum_{j=0}^n c^j \tau_2^j / j!$ . That is, thinking of  $c\tau_2^j / j!$  as  $(0, 0, \dots, c\tau_2^j / j!, 0, \dots, 0)$  and  $1$  as  $(1, 0, 0, \dots, 0)$  for  $1 \in C^\infty(M)$ , we are therefore lead to define  $e^{c\tau}$  by

$$\begin{aligned} e^{c\tau} = & \left( e^{c\tau_0}, 0, e^{c\tau_0}c\tau_2, 0, e^{c\tau_0}\frac{1}{2!}c^2\tau_2^2, 0, e^{c\tau_0}\frac{1}{3!}c^3\tau_2^3, 0, \right. \\ & \left. \dots, 0, e^{c\tau_0}\frac{1}{n!}c^n\tau_2^n \right) \in \Lambda M; \end{aligned} \quad (4.1)$$

compare (3.3). Now  $i(X^*)e^{c\tau_0} = 0$  (as  $e^{c\tau_0}$  is a function), and  $de^{c\tau_0} = ce^{c\tau_0}d\tau_0$ . That is, by (2.3),  $\theta(X^*)e^{c\tau_0} = ci(X^*)e^{c\tau_0}d\tau_0 = c[i(X^*)e^{c\tau_0}d\tau_0 + e^{c\tau_0}i(X^*)d\tau_0] = ce^{c\tau_0}i(X^*)d\tau_0$ , where  $\tau_0 = J(X) \Rightarrow$  (by (2.3), (3.10))  $i(X^*)d\tau_0 = -i(X^*)^2\sigma = 0 \Rightarrow \theta(X^*)e^{c\tau_0} \stackrel{(ii)}{=} 0$ . More generally,  $\theta(X^*)e^{c\tau_0}(c^j\tau_2^j)/j! = (\theta(X^*)e^{c\tau_0})(c^j\tau_2^j)/j! + e^{c\tau_0}(c^j/j!)\theta(X^*)\tau_2^j = e^{c\tau_0}(c^j/j!)\theta(X^*)\tau_2^j$  (by (ii))  $= 0$ , again by the fact that  $\theta(X^*)$  is a derivation and the fact that  $\theta(X^*)\tau_2 = -1/2\pi\sqrt{-1}\theta(X^*)\sigma$  with  $\theta(X^*)\sigma = 0$  (as observed earlier). By (4.1) we see therefore that  $\theta(X^*)e^{c\tau} = 0 \Rightarrow$

$e^{c\tau} \in \Lambda_{X^*} M$ , by (2.5). We claim moreover that  $d_{X^*,s} e^{c\tau} = 0$  for  $s = -2\pi\sqrt{-1}$ . By (3.6) and (4.1)

$$\begin{aligned} d_{X^*,s} e^{c\tau} = & (0, d\beta_0 + si(X^*)\beta_2, 0, d\beta_2 + si(X^*)\beta_4, 0, \\ & \dots, d\beta_{2n-2} + si(X^*)\beta_{2n}, 0) \end{aligned} \quad (4.2)$$

for  $\beta_{2j} = e^{c\tau_0} c^j \tau_2^j / j!$ . Using that  $d(\omega_1 \wedge \omega_2) \stackrel{(iii)}{=} d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$  for forms  $\omega_1, \omega_2$  of homogeneous degree and that  $e^{c\tau_0}, \tau_2$  are of even degree, we get  $de^{c\tau_0} \tau_2^j = de^{c\tau_0} \wedge \tau_2^j + e^{c\tau_0} \wedge d\tau_2^j$  where  $d\tau_2^j = 0$  (by (iii) since  $d\tau_2 = -1/(2\pi\sqrt{-1})d\sigma = 0$ )  $\Rightarrow d\beta_{2j} = (c^j/j!)e^{c\tau_0} d\tau_0 \wedge \tau_2^j \stackrel{(iv)}{=} -(c^{j+1}/j!)e^{c\tau_0}(i(X^*)\sigma) \wedge \tau_2^j$ , by (3.10). Similarly  $i(X^*)e^{c\tau_0} \tau_2^j = (i(X^*)e^{c\tau_0})\tau_2^j + e^{c\tau_0}i(X^*)\tau_2^j = e^{c\tau_0}i(X^*)\tau_2^j$ , where  $i(X^*)\tau_2^j = j\tau_2^{j-1} \wedge i(X^*)\tau_2$  (since  $i(X^*)$  also satisfies the derivative property (iii)), and since  $i(X^*)\tau_2$  and  $\tau_2$  commute as  $\deg \tau_2 = 2 \Rightarrow i(X^*)\beta_{2j}e^{c\tau_0}(c^j/(j-1)!) \tau_2^{j-1} \wedge i(X^*)\tau_2 = e^{c\tau_0}(c^j/(j-1)!) \tau_2^{j-1} \wedge i(X^*)\sigma/s$  (for  $s = -2\pi\sqrt{-1}$ )  $\Rightarrow si(X^*)\beta_{2j+2} \stackrel{v}{=} e^{c\tau_0}(c^{j+1}/j!) \tau_2^j \wedge i(X^*)\sigma$ . That is, by (iv) and (v),  $d\beta_{2j} + si(X^*)\beta_{2j+2} = 0$  (again as  $i(X^*)\tau_2$  and  $\tau_2$  commute), which by (4.2) establishes the claim. Hence the following is proved.

**Theorem 4.1.** *Suppose  $J : \mathfrak{g} \rightarrow C^\infty(M)$  which satisfies (3.10) is given, where  $\sigma$  is a symplectic structure on  $M$ . Recall that for  $X \in \mathfrak{g}$ , equation (3.11) defines a cocycle  $\tau^X \in Z(M, X^*, -2\pi\sqrt{-1})$ . Similarly for  $c \in \mathbb{C}$ , define  $e^{c\tau^X}$  by (4.1):*

$$\begin{aligned} e^{c\tau^X} = & \left( e^{cJ(X)}, 0, e^{cJ(X)} c \left( \frac{\sigma}{-2\pi\sqrt{-1}} \right), 0, e^{cJ(X)} \frac{c^2}{2!} \left( \frac{\sigma}{-2\pi\sqrt{-1}} \right)^2, 0, \right. \\ & \left. \dots, 0, e^{cJ(X)} \frac{c^n}{n!} \left( \frac{\sigma}{-2\pi\sqrt{-1}} \right)^n \right) \in \Lambda M, \end{aligned} \quad (4.3)$$

for  $\dim M = 2n$ . Then also  $e^{c\tau^X} \in Z(M, X^*, -2\pi\sqrt{-1})$ , and thus we have the cohomology class  $[e^{c\tau^X}] \in H(M, X^*, -2\pi\sqrt{-1})$ ; see (2.4), (2.6), (3.1).

## 5. THE DUISTERMAAT-HECKMAN FORMULA

Theorem 4.1 contains the basic assumption that a function  $J : \mathfrak{g} \rightarrow C^\infty(M)$  exists which satisfies condition (3.10). As pointed out earlier this assumption amounts to the assumption that the action of  $G$  on  $M$  is Hamiltonian - a point which we will now explain.

Given the symplectic structure  $\sigma$  on  $M$  there is a duality  $Y \leftrightarrow \beta_Y$  between smooth vector fields  $Y \in VM$  and smooth 1-forms  $\beta_Y \in \Lambda^1 M$  on  $M$ :



$$\beta_Y(Z) = \sigma(Y, Z) \quad \text{for every } Z \in VM. \quad (5.1)$$

$Y \in VM$  is called a Hamiltonian vector field if  $\beta_Y$  is exact:  $\beta_Y = d\phi$  for some  $\phi \in C^\infty(M)$ . Let  $HVM$  denote the space of Hamiltonian vector fields on  $M$ . Actually  $HVM$  is a Lie algebra. For example, given any  $\phi \in C^\infty(M)$ , the smooth 1-form  $d\phi$  corresponds (by the aforementioned duality) to a smooth vector field  $Y_\phi$  on  $M$ . Thus  $Y_\phi \in HVM$  and by (2.2) and (5.1) we have for every  $Z \in VM$ ,  $(i(Y_\phi)\sigma)(Z) = \sigma(Y_\phi, Z) = d\phi(Z) \Rightarrow$

$$d\phi = i(Y_\phi)\sigma. \quad (5.2)$$

The equation

$$[\phi_1, \phi_2] = Y_{\phi_1}\phi_2 \quad \text{for } \phi_1, \phi_2 \in C^\infty(M) \quad (5.3)$$

defines the Poisson bracket  $[\cdot, \cdot]$  on  $C^\infty(M)$  which converts  $C^\infty(M)$  into a Lie algebra such that the map  $\wp : \phi \rightarrow Y_\phi : C^\infty(M) \rightarrow HVM$  is a Lie algebra homomorphism; i.e.  $[Y_{\phi_1}, Y_{\phi_2}] = Y_{[\phi_1, \phi_2]}$ . The (left) action of  $G$  on  $M$  is called symplectic if  $X^* \in HVM$ ,  $\forall X \in \mathfrak{g}$ ; see (3.1). Now the map  $X \rightarrow X^* : \mathfrak{g} \rightarrow VM$  is not a Lie algebra homomorphism since  $[X_1, X_2]^* = -[X_1^*, X_2^*]$  for  $X_1, X_2 \in \mathfrak{g}$ . If we define  $\eta : \mathfrak{g} \rightarrow VM$  by  $\eta(X) = (-X^*) = -X^*$  then we do obtain a homomorphism:  $\eta([X_1, X_2]) = -[X_1, X_2]^* = [X_1^*, X_2^*] = [-\eta(X_1), -\eta(X_2)] = [\eta(X_1), \eta(X_2)]$ . In other words if the action of  $G$  is symplectic then  $\eta : \mathfrak{g} \rightarrow HVM$  is a Lie algebra homomorphism. The (left) action of  $G$  on  $M$  is called Hamiltonian if it is symplectic and if the Lie algebra homomorphism  $\eta : \mathfrak{g} \rightarrow HVM$  has a lift to  $C^\infty(M)$  - i.e. if there exists a Lie algebra homomorphism  $J : \mathfrak{g} \rightarrow C^\infty(M)$  such that the diagram

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{\wp} & HVM \\ & \nwarrow J \quad \nearrow \eta & \\ & \mathfrak{g} & \end{array} \quad (5.4)$$

is commutative:  $\eta = \wp \circ J$ , or

$$-X^* = Y_{J(X)} \quad \text{for every } X \in \mathfrak{g}. \quad (5.5)$$

We note that such a  $J$  will indeed satisfy condition (3.10). Namely, by (5.2) and (5.5),  $dJ(X) = i(Y_{J(X)})\sigma = -i(X^*)\sigma$  for  $X \in \mathfrak{g}$ . The triple

$(M, \sigma, J)$ , for  $J$  subject to (5.4), is called a Hamiltonian  $G$ -space [15, 29]. The basic example of a Hamiltonian  $G$ -space is that of an orbit  $\mathcal{O}$  in the dual space  $\mathfrak{g}^*$  of  $\mathfrak{g}$  under the adjoint action of  $G$  on  $\mathfrak{g}^*$ , where  $\sigma$  is chosen as the Kirillov symplectic form on  $M = \mathcal{O}$ , and where  $J$  is given by a canonical construction (see Appendix).

We are now in position to state the Duistermaat-Heckman formula - in a form directly derivable from Theorem 3.1.

**Theorem 5.1.** *Suppose as above that  $(M, \sigma, J)$  is a Hamiltonian  $G$ -space where  $G$  and  $M$  are compact. Orient  $M$  by the Liouville form  $\omega_\sigma$  in (3.9). Then for  $c \in \mathbb{C}$  and for  $X \in \mathfrak{g}$  with  $X^*$  non-degenerate, we have*

$$\int_M e^{cJ(X)} \omega_\sigma = \left( \frac{2\pi}{c} \right)^n \sum_{\substack{p \in M, \\ p = \text{a zero of } X^*}} \frac{e^{cJ(X)(p)}}{[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}}}. \quad (5.6)$$

Here, as in Theorem 3.1, some  $G$ -invariant Riemannian metric  $\langle, \rangle$  on  $M$  has been selected, and the square-root in (5.6) is that in (3.2).

The proof of (5.6) is quite simple, given Theorem 3.1. Namely, given the lifting  $J$  (where we have noted that (5.4) implies (3.10)) let  $c_J(X) = [e^{c\tau^X}]$  be the cohomology class constructed in Theorem 4.1, for  $c \in \mathbb{C}$ ,  $X \in \mathfrak{g}$ . By (3.7) and (4.3)

$$p^* c_J(X) = e^{cJ(X)(p)} \quad \text{for } X_p^* = 0, \quad (5.7)$$

and by (3.5) and (4.3)

$$\int_M c_J(X) = \left( \frac{c}{-2\pi\sqrt{-1}} \right)^n \int_M e^{cJ(X)} \frac{\sigma^n}{n!} = (-1)^{\frac{n}{2}} \left( \frac{c}{2\pi} \right)^n \int_M e^{cJ(X)} \omega_\sigma. \quad (5.8)$$

On the other hand given that  $X^*$  is non-degenerate, the localization formula (3.8) gives

$$\int_M c_J(X) = (-1)^{\frac{n}{2}} \sum_{\substack{p \in M, \\ p = \text{a zero of } X^*}} \frac{e^{cJ(X)(p)}}{[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}}}, \quad (5.9)$$

by (5.7). That is, by (5.8) and (5.9) we obtain exactly formula (5.6), as desired.

Note that for  $X \in \mathfrak{g}$ ,  $Z \in VM$ , and  $p \in M$ ,  $dJ(X)_p(Z_p) = [dJ(X)(Z)](p) = [(-i(X^*)\sigma)(Z)](p)$  (as  $J$  satisfies (3.10))  $= -\sigma(X^*, Z)(p)$  (by (2.2))  $= -\sigma_p(X_p^*, Z_p)$ . Hence  $dJ(X)_p = 0$  if  $X_p^* =$

0, and conversely  $dJ(X)_p = 0 \Rightarrow X_p^* = 0$  since  $\sigma_p$  is non-degenerate. (5.6) can therefore be expressed as

$$\int_M e^{cJ(X)} \omega_\sigma = \left(\frac{2\pi}{c}\right)^n \sum_{\substack{p \in M, \\ p = \text{a critical point of } J(X)}} \frac{e^{cJ(X)(p)}}{[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}}}, \quad (5.10)$$

where the critical points of  $J(X)$  are those where  $dJ(X)$  vanishes. Recall that the asymptotic behaviour of an oscillatory integral

$$I(f, t) = \int_{X(\text{=some space})} e^{\sqrt{-1}tf(x)} dx \quad (5.11)$$

for large  $t$  is given by the stationary-phase approximation - the dominant terms of this approximation being governed by the critical points of the phase  $f(x)$ . If we choose  $c = \sqrt{-1}t$ , for  $t \in \mathbb{R}$ , in (5.10), in particular, we see that the D-H formula can be viewed as an exactness result in a stationary-phase approximation of the integrals  $\int_M e^{\sqrt{-1}tJ(X)} \omega_\sigma$ , as our remarks of Section 1 indicated.

For extended and much broader discussions of material introduced here, the two references [6, 25] are especially recommended. The reference [25] in particular serves as a vast source of information for the needs of physicists. Further reading of interest is found in the references [2, 11, 13, 14, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28].

## 6. APPENDIX

The D-H formula of Theorem 5.1 was stated in the context of a Hamiltonian  $G$ -space  $(M, \sigma, J)$ . We pointed out that the premier example of such a space is an orbit  $\mathcal{O}$  in the dual space  $\mathfrak{g}^*$  of the Lie algebra  $\mathfrak{g}$  of a Lie group  $G$ , where the action of  $G$  on  $\mathfrak{g}^*$  (which is called the co-adjoint action) is induced by the adjoint action of  $G$  on  $\mathfrak{g}$ . Namely for a linear functional  $f$  on  $\mathfrak{g}$ ,  $f \in \mathfrak{g}^*$ ,

$$(a \cdot f)(X) = f(Ad(a^{-1})X) \quad \text{for } a \in G, X \in \mathfrak{g}. \quad (A.1)$$

We shall recall how the (well-known) symplectic structure  $\sigma$  on  $\mathcal{O}$  is obtained (due to A.A. Kirillov) and how the lifting  $J$  is canonically constructed. Thus we exhibit  $(\mathcal{O}, \sigma = \sigma_{\mathcal{O}}, J = J_{\mathcal{O}})$  as a key example of a Hamiltonian  $G$ -space. For this purpose it is convenient to regard the orbit of  $f$  as a homogeneous space:  $\mathcal{O} \simeq G/G_f$  where  $G_f$  is the stabilizer of  $f$ :

$$G_f = \{a \in G | a \cdot f = f\}. \quad (A.2)$$

$G_f$  is a closed subgroup of  $G$  with Lie algebra  $\mathfrak{g}_f$  given by

$$\mathfrak{g}_f = \{X \in \mathfrak{g} \mid f([X, Y]) = 0 \quad \forall Y \in \mathfrak{g}\}. \quad (\text{A.3})$$

Let  $\tau^f$  be the corresponding Maurer - Cartan 1-form on  $G$ . That is,  $\tau^f \in V^1G$  is the unique left-invariant 1-form on  $G$  subject to the condition

$$\tau^f(X)(1) = f(X) \quad \forall X \in \mathfrak{g}. \quad (\text{A.4})$$

Let  $\pi : G \rightarrow G/G_f$  denote the quotient map.

**Theorem A.1.**  *$G/G_f$  has a symplectic structure  $\sigma$  which is uniquely given by  $\pi^*\sigma = d\tau^f$ .*

Here  $\pi^*\omega_1$  denotes the pull-back of a form  $\omega_1$ . The form  $\sigma$  is also left-invariant; i.e.  $\ell_a^*\sigma = \sigma$  where  $\ell_a : G/G_f \rightarrow G/G_f$  denotes left translation by  $a \in G$ . Given  $X \in \mathfrak{g}$  define  $\phi_X : G/G_f \rightarrow \mathbb{R}$  by

$$\phi_X(aG_f) = f(\text{Ad}(a^{-1})X) = (a \cdot f)(X) \quad (\text{A.5})$$

for  $a \in G$ ;  $\phi_X$  is well-defined by (A.2). One can show by computation that

$$d\phi_X = -i(X^*)\sigma. \quad (\text{A.6})$$

That is, by (5.1),  $\beta_{-X^*} = d\phi_X \Rightarrow -X^*$  (or  $X^*$ ) is Hamiltonian for each  $X \in \mathfrak{g}$ ; i.e. the action of  $G$  on  $G/G_f$  is symplectic. To see that this action is Hamiltonian we must construct a lift  $J : \mathfrak{g} \Rightarrow C^\infty(G/G_f)$  of  $\eta : X \rightarrow -X^*$ . Namely define  $J$  by

$$J(X) = \phi_X \quad \text{for } \phi_X \text{ in (A.5)}. \quad (\text{A.7})$$

Recall that  $\wp : C^\infty(M) \rightarrow HVM$  is given by  $\wp(\phi) = Y_\phi$ . That is, by (5.2) and (A.6),  $\wp(\phi_X) = -X^* = \eta(X)$ , which shows that  $J$  does satisfy the commutative diagram in (5.4). The final step is to show that  $J$  is a homomorphism. Let  $X_1, X_2 \in \mathfrak{g}$ ,  $a \in G$ . The Poisson bracket is given by (5.3):

$$\begin{aligned} [J(X_1), J(X_2)](\pi(a)) &= (Y_{J(X_1)}J(X_2))(\pi(a)) \\ &= (\wp(J(X_1))J(X_2))(\pi(a)) = (\eta(X_1)J(X_2))(\pi(a)) \\ &(\text{again by (5.4)}) = ((-X_1^*)\phi_{X_2})(\pi(a)) \quad (\text{by (A.7)}) \\ &= \frac{d}{dt}\phi_{X_2}((\exp(-tX_1)) \cdot \pi(a))|_{t=0} \quad (\text{by (3.1)}) \\ &= \frac{d}{dt}\phi_{X_2}(\pi((\exp(-tX_1)) \cdot a))|_{t=0} \end{aligned}$$

$$\begin{aligned}
&= \frac{d}{dt} f \left( \text{Ad}(a^{-1} \exp(X_1)) X_2 \right) |_{t=0} \quad (\text{by (A.5)}) \\
&= \frac{d}{dt} f \left( \text{Ad}(a^{-1}) \text{Ad}(\exp(X_1)) X_2 \right) |_{t=0} \\
&= \frac{d}{dt} (a \cdot f) \left( \text{Ad}(\exp(X_1)) X_2 \right) |_{t=0} \quad (\text{by (A.5)}) \\
&= (a \cdot f) ([X_1, X_2]) = f \left( \text{Ad}(a^{-1}) [X_1, X_2] \right). \tag{A.8}
\end{aligned}$$

On the other hand

$$\begin{aligned}
J([X_1, X_2]) (\pi(a)) &= \phi_{[X_1, X_2]}(\pi(a)) \quad (\text{by (A.7)}) \\
&= f \left( \text{Ad}(a^{-1}) [X_1, X_2] \right) \quad (\text{by (A.5)}) \tag{A.9}
\end{aligned}$$

which proves that  $[J(X_1), J(X_2)] = J([X_1, X_2])$ .

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DEPARTAMENTO DE FISICA, UNIVERSIDADE ESTADUAL DE LONDRINA, CAIXA POSTAL 6001, LONDRINA-PARANA, BRAZIL *E-mail address:* abyts@fisica.uel.br

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01003, USA *E-mail address:* williams@math.umass.edu